Identity
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$\forall n \in \mathbb{N}$, prove that $2^n \prod_{k=1}^{n} \sin \frac{k\pi}{2n+1} = \sqrt{2n+1}$

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Let $p_n := 2^n \prod_{k=1}^{n} \sin \frac{k\pi}{2n+1}$. Note that $p_n = \sqrt{2n+1} \iff p_n^2 = 2n+1$ and

$p_n^2 = 2^{2n} \prod_{k=1}^{n} \sin^2 \frac{k\pi}{2n+1} = (-1)^n 2^{2n} \prod_{k=1}^{n} \left(0 - \sin^2 \frac{k\pi}{2n+1}\right) = P_n(0),$

where $P_n(x) := (-1)^n 2^{2n} \prod_{k=1}^{2n} \left(x^2 - \sin^2 \frac{k\pi}{2n+1}\right)$.

Let $Q_n := \frac{\sin(2n+1)t}{\sin t}, n \in \mathbb{N} \cup \{0\}$. Then Q_n satisfy to recurrence

$$Q_{n+1} + Q_{n-1} = 2(1 - 2\sin^2 t)Q_n.$$

Indeed, since $\sin(n+1)t + \sin(n-1)t = 2\cos t \sin nt, n \in \mathbb{N}$ then
$\sin(n+2)t + \sin(n-2)t + 2\sin nt = 2\cos t(\sin(n+1)t + \sin(n-1)t) =$
$4\cos^2 t \sin nt \iff \sin(n+2)t + \sin(n-2)t = (4\cos^2 t - 2)\sin nt \iff$
$\sin(n+2)t + \sin(n-2)t = 2(1 - 2\sin^2 t)\sin nt$ and, therefore,
$\frac{\sin(2n+3)t}{\sin t} + \frac{\sin(2n+1)t}{\sin t} = 2(1 - 2\sin^2 t)\frac{\sin(2n-1)t}{\sin t}$.

Denoting $x := \sin t$ we obtain $Q_{n+1} + Q_{n-1} = 2(1 - 2x^2)Q_n, n \in \mathbb{N}$ and $Q_0 = 1$,
$Q_1 = \frac{\sin 3t}{\sin t} = 3 - 4\sin^2 t = 3 - 4x^2$.

Thus, Q_n in fact is polynomial $Q_n(x)$ of degree $2n$.

Let q_n is coefficient of x^{2n} in $Q_n(x)$. Then $q_0 = 1, q_1 = -2^2$ and recurrence $Q_{n+1}(x) + Q_{n-1}(x) = 2(1 - 2x^2)Q_n(x)$ implies recurrence $q_{n+1} = -4q_n, n \in \mathbb{N}$.

Hence, $q_n = (-1)^n 2^{2n}$ and since $P_n(x)$ and $Q_n(x)$ have the same roots
$(Q_n(x) = 0 \iff \left(\frac{\sin(2n+1)t}{\sin t} = 0\right) \wedge (x = \sin t) \iff$

$x_k = \sin \frac{k\pi}{2n+1}, k = \pm 1, \pm 2, \ldots, \pm n)$

then $P_n(x) = Q_n(x)$. Let $c_n := Q_n(0), n \in \mathbb{N}$. Since $c_{n+1} + c_{n-1} = 2c_n, n \in \mathbb{N}$ and $c_0 = 1, c_1 = 3$ then $c_n = 2n+1$ and, therefore, $p_n^2 = 2n+1$.